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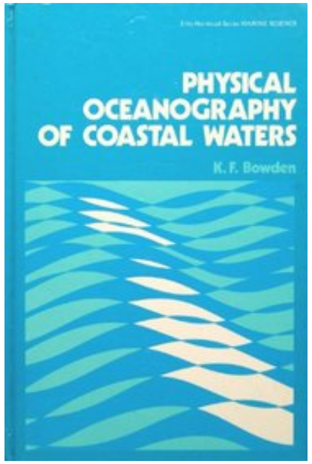
Surface gravity (wind) waves
Part 1


Main references



K. F. Bowden.
Physical Oceanography of coastal waters:
Chapter 3: Surface waves
Section 3.1, 3.2

Surface waves characteristics

The main characteristics of surface waves can be summarised as follows:

- They are of relatively short period (1 to 30 s).
- In deep water their influence is restricted to a comparatively shallow layer (small amplitude)
- Vertical and horizontal displacement have similar magnitude
- Vertical acceleration is significant and comparable with gravity
- Vertical and horizontal accelerations are large compared with the components of geostrophic acceleration.

Therefore……….

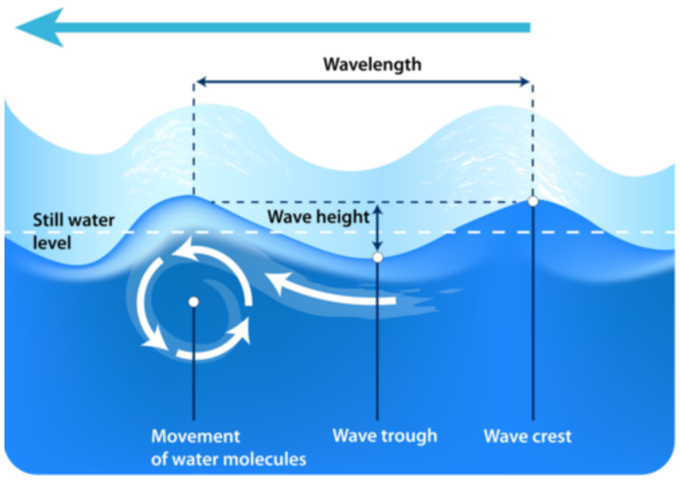
Surface waves characteristics

Vertical and horizontal accelerations are large compared with the components of geostrophic acceleration.

Surface gravity waves dynamics is a large Rossby number (Ro) $Ro = \dfrac{U}{Lf}$ where U, and L are the characteristics velocity and length scale of the motion and f is the Coriolis parameter. Hence the Coriolis force has a negligible influence.

Vertical and horizontal displacement have similar magnitude

The hydrostatic approximation does not hold anymore.



Equations for surface gravity waves dynamics

The equation system to be considered is:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+F_x$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+F_y$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}-g+F_z$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Where (as usual) $F_{(x,y,z)}$ indicate all the forces, different from pressure forces acting on the fluid.
Some simplifications can be made……………

Equations for surface gravity waves dynamics

Some simplifications can be made……………

Motion can be considered limited to only one horizontal direction (x). Then $v=0$ and $\frac{\partial}{\partial y}=0$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\cancel{v\frac{\partial u}{\partial y}}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+F_x$$

$$\cancel{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+F_y}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\cancel{v\frac{\partial w}{\partial y}}+\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g+F_z$$

$$\frac{\partial u}{\partial x}+\cancel{\frac{\partial v}{\partial y}}+\frac{\partial w}{\partial z}=0$$

Equations for surface gravity waves dynamics

Some simplifications can be made……………
Waves are considered of relatively small amplitude, so that the non linear (higher order acceleration) terms can be neglected

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x$$

$$\frac{\partial w}{\partial t} + \cancel{u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + F_z$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Equations for surface gravity waves dynamics

Some simplifications can be made……………

Frictional terms (eddy viscosity) is assumed negligible, then $F_x = F_z = 0$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \cancel{F_x} \\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + \cancel{F_z} \\ \frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0 \end{aligned}}$$

Equations for surface gravity waves dynamics

So that the final equation system is:

$$\boxed{\begin{aligned}&\frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}\\&\frac{\partial w}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g\\&\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\end{aligned}}$$

at the free surface $(z=\eta)$ two boundary conditions applies:

$$\boxed{w\big|_{z=\eta}=\frac{\partial \eta}{\partial t}}$$

$$\boxed{p\big|_{z=\eta}=p_a}$$

Where p_a= atmospheric pressure (hereafter considered constant in space and time)

Equations for surface gravity waves dynamics

A further assumption made is that the waves are "irrotational": i.e. they have no vorticity, and the curl of the velocity vector is zero.

The condition for irrotationality (motion confined in the x-z plane) is: $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$

Differentiating with respect to:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad \Rightarrow \quad x \quad \Rightarrow \quad \frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g \quad \Rightarrow \quad z \quad \Rightarrow \quad \frac{\partial}{\partial z}\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad t \quad \Rightarrow \quad \frac{\partial}{\partial t}\frac{\partial u}{\partial x} + \frac{\partial}{\partial t}\frac{\partial w}{\partial z} = 0$$

Equations for surface gravity waves dynamics

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial}{\partial z}\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z^2}$$

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial x} + \frac{\partial}{\partial t}\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial x} - \frac{1}{\rho_0}\frac{\partial^2 p}{\partial z^2} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial w}{\partial z} = -\frac{\partial}{\partial t}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0$$

Small amplitude progressive waves

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2}=0$$

A solution of the above equation is sought for an harmonic wave travelling in the x direction. The following form of the solution is assumed:

$$p = p_a - \rho g z + Z(z)\cos\left(\kappa x - \sigma t\right)$$

The first two terms on the R.H.S. give the pressure at depth z under hydrostatic conditions, while the 3rd term describes a pressure wave of wavelength:

$$\lambda = 2\pi / \kappa$$

And period:

$$T = 2\pi / \sigma$$

With an amplitude $Z(z)$ (depth dependent).

Small amplitude progressive waves

Entering: $p = p_a - \rho g z + Z(z)\cos\left(\kappa x - \sigma t\right)$ into $\dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial z^2} = 0$

One find that $Z(z)$ must satisfy the differential equation:

$$\frac{\partial^2 Z}{\partial z^2} - \kappa^2 Z = 0$$

Whose solution is

$$Z\left(z\right) = Ae^{\kappa z} - Be^{-\kappa z}$$

A,B=integration constants

It has to be distinguished now if the wave is travelling in deep or in finite depth water.

Small amplitude progressive waves

In deep water:

The effect of surface waves becomes negligible at great depth. Then

$$Z(z)=0 \quad \text{for} \quad z \to -\infty$$

As a consequence in

$$Z(z) = Ae^{\kappa z} - Be^{-\kappa z}$$

We have $B=0$ and the solution for p becomes:

$$p = p_a - \rho g z + Ae^{\kappa z}\cos(\kappa x - \sigma t)$$

Applying the boundary conditions defined above for $z=\eta$ we can find that κ and σ have to satisfy:

$$\sigma^2 = g\kappa$$

Since the propagation velocity c is given by: $c = \dfrac{\lambda}{T} = \dfrac{\sigma}{\kappa}$

we arrive at: $c^2 = \dfrac{g}{\kappa} = \dfrac{g\lambda}{2\pi}$

Small amplitude progressive waves

In deep water:

By posing $a=A/\varrho g$ and using:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g$$

$$w\big|_{z=\eta} = \frac{\partial \eta}{\partial t}$$

$$p = p_a - \rho g z + A e^{\kappa z}\cos(\kappa x - \sigma t)$$

the following is obtained:

$$\eta = a\cos(\kappa x - \sigma t)$$

$$u = \sigma a e^{\kappa z}\cos(\kappa x - \sigma t)$$

$$w = \sigma a e^{\kappa z}\sin(\kappa x - \sigma t)$$

$$p = p_a - \rho g z + \rho g a e^{\kappa z}\cos(\kappa x - \sigma t)$$

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In deep water

$$\eta = a\cos(\kappa x - \sigma t)$$

$$u = \sigma a e^{\kappa z}\cos(\kappa x - \sigma t)$$

$$w = \sigma a e^{\kappa z}\sin(\kappa x - \sigma t)$$

$$p = p_a - \rho g z + \rho g a e^{\kappa z}\cos(\kappa x - \sigma t)$$

a: amplitude of the vertical displacement of the sea surface

u,w: horizontal and vertical vertical components of the particle velocity at the depth z.

The horizontal and vertical displacements of a water particle (η_x, η_z respectively) can be determined
By:

$$u = \frac{\partial \eta_x}{\partial t} \quad \Rightarrow \quad \eta_x = -a e^{\kappa z}\sin(\kappa x - \sigma t)$$

$$w = \frac{\partial \eta_z}{\partial t} \qquad\qquad \eta_z = a e^{\kappa z}\cos(\kappa x - \sigma t)$$

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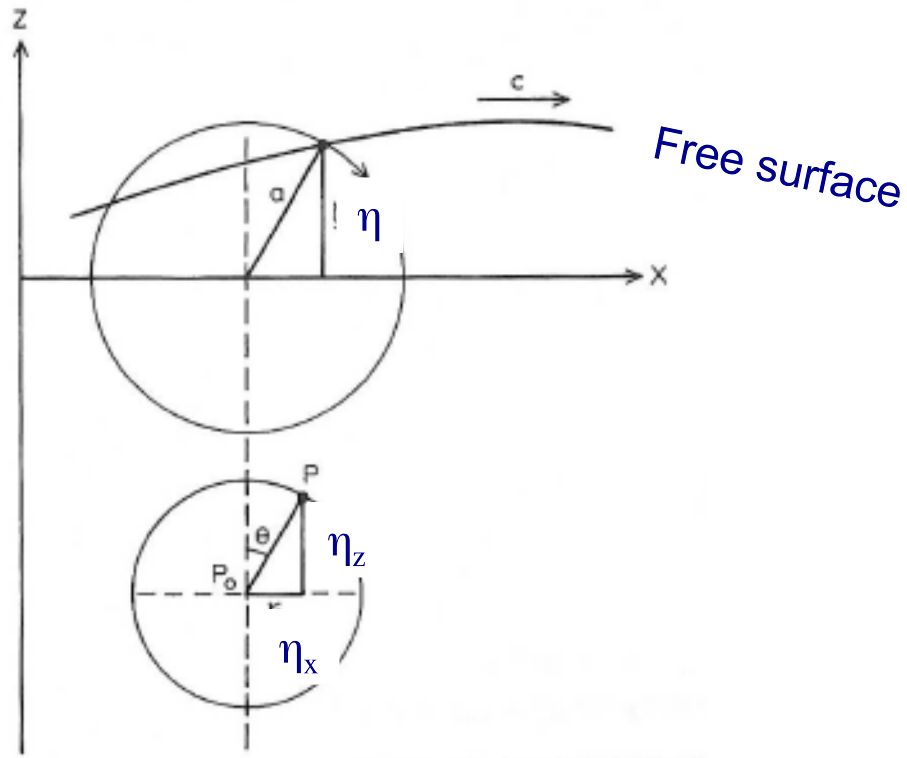
In deep water

$$\eta_x = -ae^{\kappa z}\sin\left(\kappa x - \sigma t\right) \qquad \eta_z = ae^{\kappa z}\cos\left(\kappa x - \sigma t\right)$$

The equations above indicate that water particles move in circular orbits of radius $ae^{\kappa z}$



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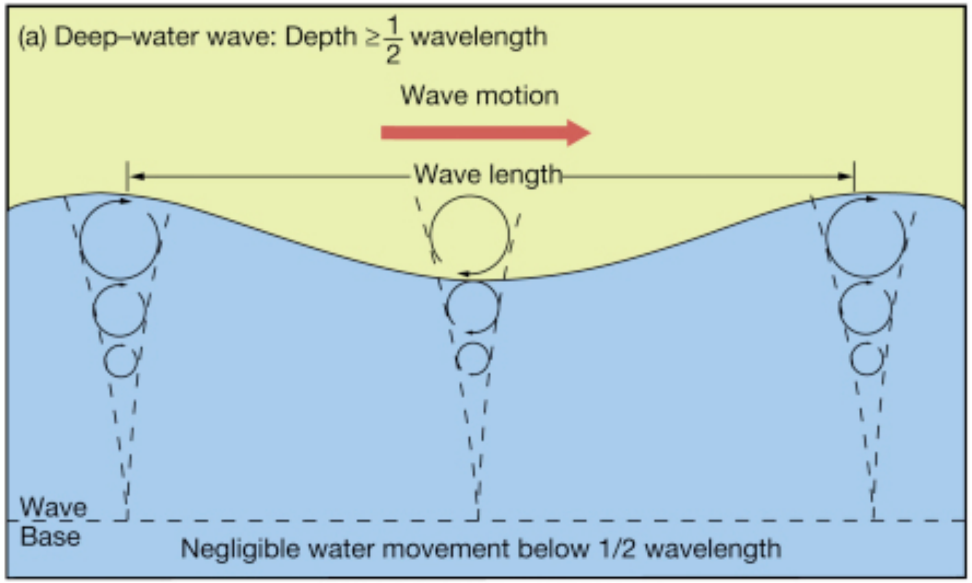
In deep water

The displacement, the particle velocity and the oscillatory part of the pressure, all decrease exponentially with increasing depth.

The attenuation factor $e^{\kappa z} = \exp\left(2\pi z / \lambda\right)$

At $z= \lambda/2$ this factor becomes $\exp\left(-\pi\right) = 0.043$ i.e. wave motion is reduced to 1/23 (4%) of the surface amplitude.



Small amplitude progressive waves

Water of finite depth

water depth not large compared with wavelength.

The bottom boundary boundary condition becomes: $$\left. w \right|_{z=-H} = 0$$

Equation:

$$Z(z) = Ae^{\kappa z} - Be^{-\kappa z}$$

Holds entirely. Inserting the solution for Z(z) is inserted into into

$$p = p_a - \rho g z + Z(z)\cos(\kappa x - \sigma t)$$

And w is found using

$$\boxed{\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g}$$

Applying the bottom and surface boundary conditions the following set of equations is determined:

Small amplitude progressive waves

Water of finite depth

Applying the bottom and surface boundary conditions the following set of equations is determined:

$$\eta = a\cos(\kappa x - \sigma t)$$

$$c^2 = \frac{g}{\kappa}\tanh\kappa H$$

$$p = p_a + \rho g z + \rho g a\frac{\cosh(H+z)}{\cosh\kappa H}$$

$$w = \sigma a\frac{\sinh\kappa(H+z)}{\sinh\kappa H}\sin(\kappa x - \sigma t)$$

$$\eta_x = -a\frac{\cos\kappa(H+z)}{\sinh\kappa H}\sin(\kappa x - \sigma t)$$

$$\eta_z = a\frac{\sin\kappa(H+z)}{\sinh\kappa H}\sin(\kappa x - \sigma t)$$

Small amplitude progressive waves

Water of finite depth

$$c^2 = \frac{g}{\kappa}\tanh \kappa H \quad \Rightarrow \quad c^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi H}{\lambda}$$

Velocity now depends on depth and on wavelength.

Letting H>>λ then $\tanh\frac{2\pi H}{\lambda} \to 1$

The velocity for waves on deep water is recovered

Letting H<<λ then $\tanh\frac{2\pi H}{\lambda} \to \frac{2\pi H}{\lambda}$

And $c^2=gH$ The"long wave " velocity is recovered.

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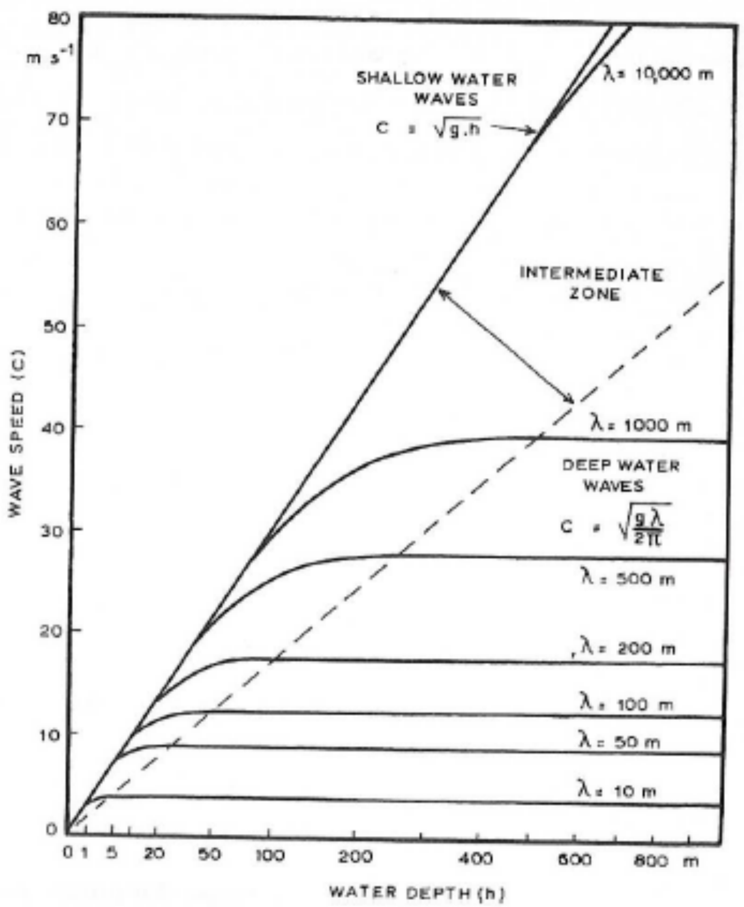
Water of finite depth



In general:

$\lambda/H<2$: deep water waves equation may be used

$$c^2 = \frac{g}{\kappa} = \frac{g\lambda}{2\pi}$$

$\lambda/H>20$ the long wave equation may be used

$$c^2=gH$$

$2\lambda<\lambda/H<20$ the complete equation must be used

$$c^2 = \frac{g}{\kappa}\tanh \kappa H$$

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Water of finite depth

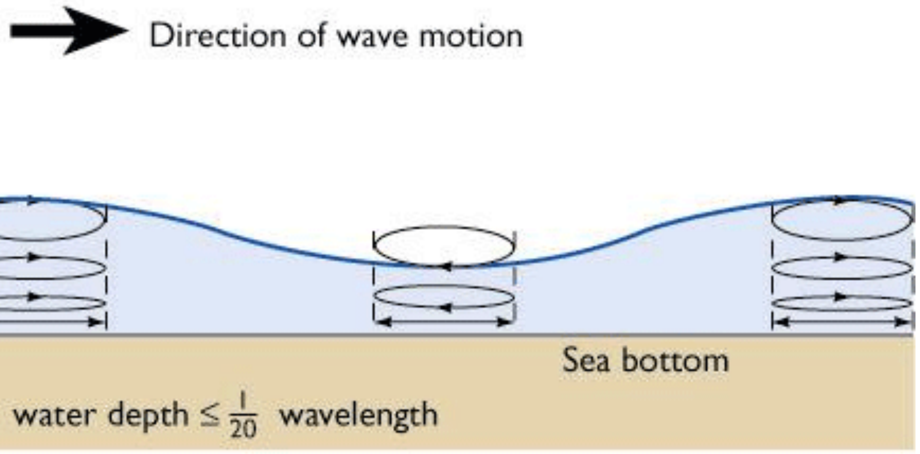
Equations for horizontal and vertical; displacement:

$$\eta_x = -a\frac{\cos\kappa(H+z)}{\sinh\kappa H}\sin(\kappa x-\sigma t)$$

$$\eta_z = a\frac{\sin\kappa(H+z)}{\sinh\kappa H}\sin(\kappa x-\sigma t)$$

Indicate that the orbits are no more round but elliptical (major axis horizontal)

As the bottom is approached the ellipses become flatter and the water particles moves back and forth horizontally



(b) SHALLOW-WATER WAVE